2. Substitution and Simplifying

Try $u = \sqrt{x}$, u = inside, u = e^x, u = trig.

Completing the square.

Trig facts.

Square identities, half-angle.

Triangle trick.

Random Integrals from Old Finals:

1. $\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$

OR $x = \sin\theta$

2. $\int \frac{x^2-x+8}{x^3+4x}\,dx = \int \frac{x^2-x+8}{x(x^2+4)}\,dx$

$\frac{x^2-x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

3. $\int 2x\ln(x+5)\,dx$

$u = \ln(x+5)$ $\quad dv = 2x\,dx$

$du = \frac{1}{x+5}\,dx$ $\quad v = x^2$

4. $\int \cos^3(x)\,dx = \int (1-\sin^2(x))\cos(x)\,dx$

$u = \sin(x)$

5. $\int_0^2 \frac{1}{\sqrt{x^2+2x+4}}\,dx = \int_0^2 \frac{1}{\sqrt{(x+1)^2+3}}\,dx$

$x^2+2x+1-1+4 = (x+1)^2+3$

$x+1 = \sqrt{3}\tan\theta$

6. $\int_1^3 \frac{1}{x^2+x^3}\,dx = \int_1^3 \frac{1}{x^2(1+x)}\,dx$

$\frac{1}{x^2(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+x}$

7. $\int \tan^2 x\sec^4(x)\,dx = \int \tan^2(x)(1+\tan^2(x))\sec^2(x)\,dx$

$u = \tan(x)$

8. $\int \frac{1}{(1+\sqrt{x})^3}\,dx = \int \frac{1}{t^3}\,2(t-1)\,dt$

$= 2\int t^{-2} - t^{-3}\,dt$

$t = 1+\sqrt{x} \Rightarrow (t-1)^2 = x$

$2(t-1)\,dt = dx$

9. $\int \sin(x)\sqrt{\cos(x)}\,dx = \int -\sqrt{t}\,dt$

$= -\frac{2}{3}t^{3/2} + C$

$= -\frac{2}{3}\cos^{3/2}(x) + C$

$t = \cos(x)$

$dt = -\sin(x)\,dx$

$(-1)dt = \sin(x)\,dx$

3. Improper Integrals:

a) Rewrite as a limit!!

b) Integrate

c) Take limit

4. Trapezoid/Simpson Rules

a) Set up integral, then compute width and label tickmarks.

b) Use formula.

Random Improper Integrals:

1. $\int_1^2 \frac{x}{\sqrt{x-1}}dx = \lim_{t\to 1^+} \int_t^2 \frac{x}{\sqrt{x-1}}dx$

2. $\int_{-3}^{\infty} xe^{-x}dx = \lim_{t\to\infty} \int_{-3}^{t} xe^{-x}dx$

3. $\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)}dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}(1+x)}dx$

$u=\sqrt{x}$

Approximation Example:

1. Use Simpson's Method with n = 4 subdivision to approximate the value of

$$\int_0^4 \sqrt{1+4x^4}dx$$

$$\Delta x = \frac{4-0}{4} = 1$$

$x_0 = 0$ $+\Delta x$

$x_1 = 1$ $+\Delta x$

$x_2 = 2$

$x_3 = 3$

$x_4 = 4$

$$\underbrace{\tfrac{1}{3}}_{}\overset{\Delta x}{(1)}\left[\sqrt{1+4(0)^4} + 4\sqrt{1+4(1)^4} + 2\sqrt{1+4(2)^4} + 4\sqrt{1+4(3)^4} + \sqrt{1+4(4)^4}\right]$$

5. New Applications

a) **Average value** = $\frac{1}{b-a}\int_a^b f(x)dx$

b) **Work** = $\int_a^b (Force)(Dist)$

Step 1: Draw picture (start and end)
Label clearly.
Draw a typical subdivision.

Step 2: Find pattern for *Force* and *Dist*.

Step 3: Integrate.

Type 1: "Changing force"
Force changing as object is moved
(leaky bucket, springs, given force).

f(x) = "force formula at x"
Force = f(x), *Dist* = Δx;

$$\text{Work} = \int_a^b f(x)dx$$

Type 2: "Stack of books"
(chain, pumping)

Chain/Cable:

Given *k* = density = force/length
if x = 0 is labeled at the top.
then for any subdivision that makes it to the top:
Force = k Δx, *Dist* = x

$$\text{Work} = \int_a^b k\,\text{x}\,dx$$

Pumping:

Given k = density = force/volume
if top is y = b, then
Force = k(Area)Δy, *Dist* = b – y;

$$\text{Work} = \int_a^b k(\text{Area})(b-y)dy$$

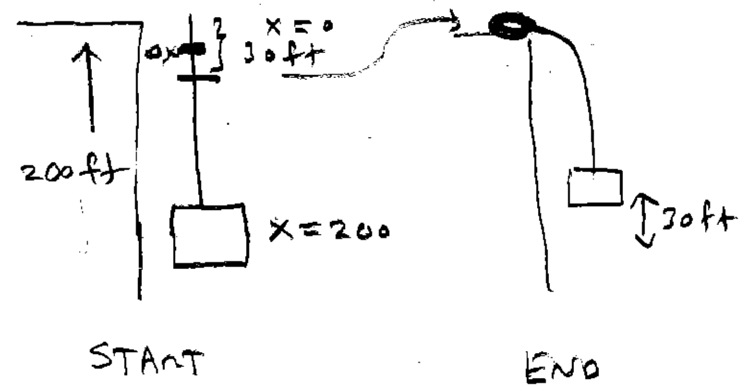
You need to have an basic understanding of how we label and find pattern to be able to adapt to changes in these problems!!

Applications from old tests:

1. Find the average value of $\cos^3(x)$ on the interval 0 to $\pi/2$.

$$\frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2}\cos^3(x)\,dx$$

3. A 1600 lb elevator is suspended by a 200 ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?



WORK TO LIFT ELEVATOR

Force = 1600 lbs

Dist = 30 ft

Work = $1600 \cdot 30$

$= 48{,}000$ ft-lbs

WORK TO LIFT CABLE FOR $0 \le x \le 30$

Upper 30 ft of cable makes it to top

Force = $10x$

Dist = x

$$\int_0^{30} x \cdot 10\,dx$$

$$5x^2 \Big|_0^{30}$$

$5(30)^2 = 4500$ ft-lbs

NOTE THE DIFFERENCE

WORK TO LIFT CABLE FOR $30 \le x \le 200$

ALL OF THIS SHIFTS UP 30 ft

Force = $10x$

Dist = 30

$$\int_{30}^{200} 30 \cdot 10\,dx$$

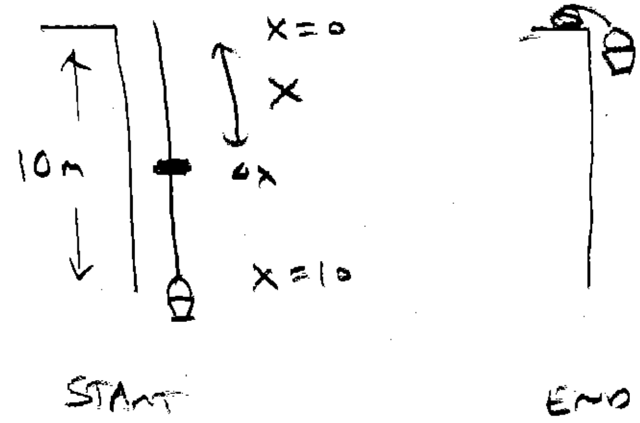
$$300x \Big|_{30}^{200}$$

$300 \cdot (200 - 30)$

$300 \cdot 170 = 51{,}000$ ft-lbs

TOTAL = 48,000 + 4,500 + 51,000 = $\boxed{103{,}500 \text{ ft-lbs}}$

4. A rope is used to pull a bucket full of water up from a well that is 10 m deep. The rope has a total mass of 5 kg. The bucket has a mass of 11 kg. Find the total work done in lifting the bucket to the top (Recall: Accel. due to gravity is 9.8 m/s^2)



MASS = 11 kg $\Rightarrow$ FORCE = $11 \cdot 9.8$ N

= 107.8 N

BUCKET

FORCE = 107.8 N

DIST = 10 m

$\Rightarrow$ WORK = $107.8 \cdot 10$

= 1,078 J

ROPE

TOTAL MASS = 5 kg $\Rightarrow$ TOTAL FORCE = $5 \cdot 9.8$ = 49 N

$\Rightarrow$ DENSITY = $\frac{49 \text{ N}}{10 \text{ m}} = 4.9 \frac{\text{N}}{\text{m}}$

FOR $0 \le x \le 10$

FORCE = $4.9 \Delta x$

DIST = x

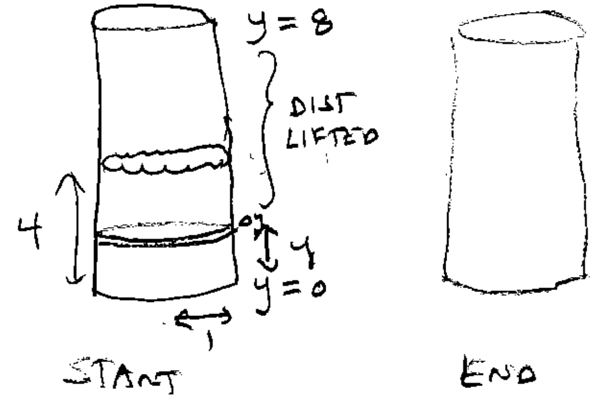
$$\int_0^{10} x \, 4.9 \, dx$$

$$\frac{4.9}{2} x^2 \Big|_0^{10}$$

$$\frac{4.9}{2} 100 = 245 \text{ J}$$

TOTAL WORK = 1078 + 245 = $\boxed{1323 \text{ J}}$

5. A well is in the shape of a cylinder of radius 1 meter and depth 8 meters. It is half full of water. Find the word to pump all the water to the top. (Recall: Water weighs 9800 N/m^3)



For $0 \le y \le 4$

$$\text{FORCE} = 9800 \cdot \overset{\text{HORIZ.}}{\text{AREA}} \cdot \Delta y$$

$$= 9800 \cdot \pi (1)^2 \Delta y = 9800\pi \Delta y$$

$$\text{DIST} = 8 - y$$

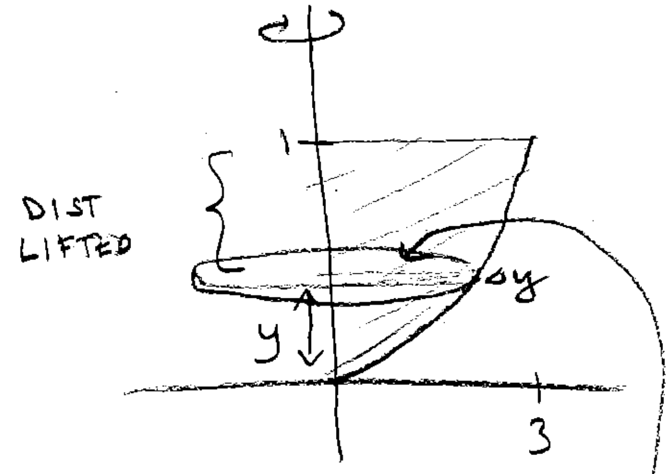
$$\text{WORK} = \int_0^4 (8-y)\, 9800\pi \, dy$$

$$= 9800\pi \left[8y - \tfrac{1}{2}y^2 \Big|_0^4 \right]$$

$$= 9800\pi \left[32 - 8 \right]$$

$$= \boxed{235{,}200\pi \text{ J} \approx 738{,}902.59 \text{ J}}$$

6. The portion of the graph $y = x^2 / 9$ between $x = 0$ and $x = 3$ is rotated about the y-axis to form a container. The container is full of a liquid that has density 100 lbs/ft³. Find the work required to pump all the liquid to the top of the container.



$$y = \frac{1}{9}x^2$$
$$\Rightarrow 9y = x^2$$
$$\Rightarrow x = 3\sqrt{y}$$

For $0 \le y \le 1$

$$\text{FORCE} = 100 \cdot \underset{\text{AREA}}{\text{HORIZ.}} \ \Delta y$$
$$= 100\, \pi (3\sqrt{y})^2 \Delta y$$
$$= 900\pi\, y\, \Delta y$$
$$\text{DIST} = 1 - y$$

$$\Rightarrow \text{WORK} = \int_0^1 (1-y)\, 900\pi\, y\, dy = 900\pi \int_0^1 y - y^2\, dy$$
$$= 900\pi \left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 \Big|_0^1 \right]$$
$$= 900\pi \left[\tfrac{1}{2} - \tfrac{1}{3} \right] = \frac{900\pi}{6}$$

$$\boxed{\begin{aligned} &= 150\pi \text{ ft-lbs} \\ &\approx 471.24 \text{ ft-lbs} \end{aligned}}$$